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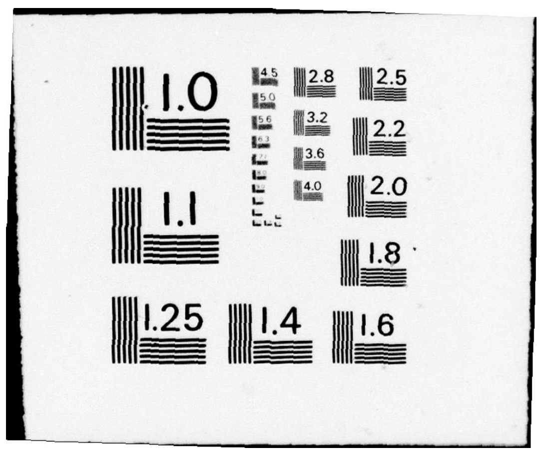
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WHITE NOISE ESTIMATORS FOR SEISMIC DATA PROCESSING IN OIL EXPLORATION

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ABSTRACT

This is basically a theoretical paper motivated by a problem from seismic data processing in oil exploration. We develop a Kalman filtering approach to obtaining optimal smoothed estimates of the so-called reflection coefficient sequence. This sequence contains important information about subsurface geometry. Our theoretical problem is one of estimating white plant noise for the systems described in Eqs. (1) and (2). By means of the equations which are derived herein, it is possible to compute fixed-interval, fixed-point, or fixed-lag optimal smoothed estimates of the reflection coefficient sequence, as well as respective error covariance-matrix information. Our optimal estimators are compared with an ad hoc "prediction error filter," which has recently been reported on in the geophysics literature. We show that one of our estimators performs at least as well as, and, in most cases better than, the prediction error filter.

INTRODUCTION

This is basically a theoretical paper motivated by a problem from seismic data processing in oil exploration. The theoretical problem is one of estimating white plant noise for models $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\left.\begin{aligned} \underline{x}(k+1) &= \Phi\underline{x}(k) + \underline{\mu}(k) \\ \underline{z}(k) &= H\underline{x}(k) + \underline{v}(k) \end{aligned}\right\} \mathcal{S}_1 \qquad (1)$$

$$\left.\begin{aligned} \underline{x}(k+1) &= \Phi\underline{x}(k) + \Gamma\underline{w}(k) \\ \underline{z}(k) &= H\underline{x}(k) + \underline{v}(k) \end{aligned}\right\} \mathcal{S}_2 \qquad (2)$$

The rest of this section briefly describes the seismic data processing problem and its relation to this theoretical problem.

One of the most useful methods to predict the presence of oil is reflection seismology. On land, for example, an explosive is detonated below the earth's surface, imparting a pulse of energy into the earth.[1] As Wood and Treitel (1) state, "This source pulse ... is split into a large number of waves traveling along various paths determined by the material properties of the medium. Whenever such a wave encounters a change in acoustic impedance (acoustic impedance is the product of rock density and rock propogation velocity), a certain fraction of the incident wave is reflected upwards. Seismic detectors situated at the earth's surface record the continual motion of the earth under the impact of seismic waves impinging from below. This recording is performed digitally at a fixed sampling increment. The resultant set of discrete observations is called a ... 'seismic trace' and constitutes a sample of a time series."

In 1954, Enders A. Robinson (4 and 5) proposed a convolution summation model to describe the signal received by a seismic sensor. We write this model, as

$$z(k) = V_R(k) + n(k) \qquad (3)$$

$$V_R(k) = \sum_{j=1}^{k} \mu(j)\, V^+(k-j) \qquad (4)$$

where $V_R(k)$ is the noise free seismic trace; n(k) is "measurement" noise which accounts for physical effects not explained by $V_R(k)$, as well as sensor noise; k is short for time t_k; $V^+(i)$, $i = 0,1,2,\ldots,I$, is a sequence associated with the basic seismic wavelet (2 and 5); and $\mu(j)$, $j = 1,2,\ldots$, is the reflection coefficient sequence. This convolution summation model can be derived from physical principles and some simplifying assumptions, such as: normal incidence (i.e., horizontal layering), each layer is homogeneous and isotropic, small strains, and, pressure and velocity (or displacement) satisfy a one-dimensional wave equation.[2] Signal $V_R(k)$, which is recorded at the earth's surface, is a superposition of

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[1] The reader interested in elements of the seismic prospecting method and the seismic reflection technique should see Anstey (2), Chs. 1 and 3, and Dix (3).

[2] It is useful, for conceptual purposes, to draw the analogy between a layered earth system and a sequence of connected lossless transmission lines. At the transmission line connections (discontinui-

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wavelets which are reflected from the interfaces of subsurface layers. The $\mu(j)$ are related to interface reflection and transmission coefficients.

The reflection coefficient sequence often has peaks at subsurface interfaces, and is often assumed to be random and uncorrelated (5). It can be represented graphically as a sequence of "spikes" which occur at $k = 1, 2, \ldots$ (Figure 1). We would like to learn as much as possible about the peak values of the $\mu(j)$ sequence, including amplitude and time of occurrence, so that information can be assembled about the subsurface geometry.

If noise was not present, and we had numerical values for sequences $V_R(i)$ and $V^+(i)$, then the reflection coefficient sequence could be reconstructed by passing $V_R(i)$ through an "inverse filter." Let $V_R(z)$, $V^+(z)$, and $\mu(z)$ denote the z-transforms of sequences $V_R(i)$, $V^+(i)$, and $\mu(i)$. Then, Eq. (2) can also be written, in terms of these z-transforms, as $V_R(z) = \mu(z)V^+(z)$, from which we see that $\mu(z) = V_R(z)(1/V^+(z))$; hence, when $V_R(z)$ is operated upon by the "inverse filter" $1/V^+(z)$, we obtain $\mu(z)$.

In the noisy situation, it is common practice to extract the reflection coefficient sequence using a technique which was developed by Robinson (4 and 5) known as predictive deconvolution. The major component of this technique is a digital Wiener filter, which requires correlation function information. The following modeling assumptions are associated with Robinson's work (1): (1) the layered earth is a linear system, (2) the basic seismic wavelet is minimum delay (i.e., minimum phase), and (3) the reflection coefficient sequence is random and uncorrelated.

Because the Wiener filtering approach to predictive deconvolution is limited by these modeling assumptions, which may not always be valid, it is important to look for alternative approaches which will permit more flexible modeling assumptions. Kalman filtering is such an alternative, and has recently begun to receive attention from geophysicists (7-9). In this paper we shall develop a Kalman filtering approach to obtaining optimal estimates of the reflection coefficient sequence.

In order to use a Kalman filter, a state space representation is required, in contrast to the convolution summation representation in Eq. (4), which is the basis for use of a Wiener filter. From linear system theory (10, for example), it is well known that the output, $y(k)$, of a linear, discrete-time, time-invariant, causal system, whose input $r(k)$ is zero prior to time zero, is

$$y(k) = \sum_{j=0}^{k} r(j)\,h(k-j) \qquad (5)$$

where $h(i)$ is the unit function response of the system. Comparing Eqs. (5) and (4), we are led to the following important system interpretation for the seismic trace model: signal $V_R(k)$ can be thought of as the output of a linear time-invariant system whose unit response is $V^+(i)$ and input sequence is the reflection coefficient sequence, $\mu(i)$. (Figure 2).

We shall assume that a state space representation of Eqs. (3) and (4) is our starting point for predictive deconvolution. How to obtain such a representation from the available information is not the subject of the present paper; but, is an important topic that will be discussed in the context of this application in a future paper.

For a single-input single-output system, a most efficient state space realization is the phase variable canonical form (11):

$$\underline{x}(k+1) = \Phi\underline{x}(k) + \underline{b}\mu(k) \qquad (6)$$

$$z(k) = \underline{h}'\underline{x}(k) + n(k) \qquad (7)$$

where $\underline{x} = \mathrm{col}(x_1, x_2, \ldots, x_n)$, $\underline{b} = \mathrm{col}(b_1, b_2, \ldots, b_n)$, $\underline{h} = \mathrm{col}(1, 0, \ldots, 0)$,

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & 0 & 0 & & & & 1 \\ -a_n & -a_{n-1} & . & . & . & . & -a_1 \end{pmatrix},$$

and $x_1(k) = V_R(k)$. Accepting Eqs. (6) and (7) as our starting point for predictive deconvolution, instead of Eqs. (3) and (4), we shall study the estimation of the "white" reflection coefficient sequence, $\mu(k)$, by means of Kalman filter theory.

Bayliss and Brigham (7) and Crump (8) have attempted to apply kalman filtering to predictive deconvolution; however, they assume $\mu(k)$ is a colored noise process and proceed along a route which is quite different from the one we shall take. Ott and Meder (9) have attempted to estimate $\mu(k)$ via Kalman filtering; however, as we point out in Section IV, their estimator is an ad hoc one, and does not perform as well as ours.

We shall imbed the features of Eqs. (6) and (7) in a more general setting; namely, we shall study the problems of estimating white plant noise for the models $\mathscr{S}_1$ and $\mathscr{S}_2$, which are given in Eqs. (1) and (2) respectively. Optimal estimators for $\underline{\omega}(k)$ and $\underline{w}(k)$ are developed in Section II. The relationship of zero estimates for $\underline{w}(k)$ to zero Markov parameters is examined in Section III. We relate our results to the Ott and Meder results in Section IV.

OPTIMAL WHITE-PLANT-NOISE ESTIMATORS

Introduction

Let us direct our attention, for the time being, at $\mathscr{S}_1$, in Eq. (1), in which $\underline{x} \in R^n$, $\underline{\omega} \in R^n$, $\underline{z} \in R^m$, $\underline{v} \in R^m$, and $\underline{\omega}(k)$ and $\underline{v}(k)$ are uncorrelated gaussian white sequences with covariances Q and

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ties), waves are reflected and transmitted, just as they are at the interfaces between earth layers.

R, respectively. Our objective is to obtain a minimum-variance estimator for the white-plant-noise sequence $\underline{\omega}(k)$.

Let

$$Z(j) = \{\underline{z}(1), \underline{z}(2), \dots, \underline{z}(j)\}, \tag{8}$$

and $\hat{\underline{x}}(k|j)$ denote the minimum variance estimate of $\underline{x}(k)$ which uses all the measurements in $Z(j)$. From a fundamental result in estimation theory (12, for example), it is well known that $\hat{\underline{x}}(k+1|k) = E\{\underline{x}(k+1)|Z(k)\}$ and $\hat{\underline{x}}(k+1|k+1) = E\{\underline{x}(k+1)|Z(k+1)\}$, and that $\hat{\underline{x}}(k+1|k)$ and $\hat{\underline{x}}(k+1|k+1)$ can be computed from the Kalman filter, whose equations are:

$$\hat{\underline{x}}(k+1|k) = \Phi\hat{\underline{x}}(k|k) \tag{9}$$

$$P(k+1|k) = \Phi P(k|k)\Phi' + Q \tag{10}$$

$$K(k+1) = P(k+1|k)H'[HP(k+1|k)H' + R]^{-1} \tag{11}$$

$$\tilde{\underline{z}}(k+1|k) = \underline{z}(k+1) - H\hat{\underline{x}}(k+1|k) \tag{12}$$

$$\hat{\underline{x}}(k+1|k+1) = \hat{\underline{x}}(k+1|k) + K(k+1)\tilde{\underline{z}}(k+1|k) \tag{13}$$

$$P(k+1|k+1) = [I - K(k+1)H]P(k+1|k) \tag{14}$$

Observe, from Eq. (1), that $\underline{z}(k+1)$ depends on $\underline{\omega}(k), \underline{\omega}(k-1), \dots, \underline{\omega}(1)$, whereas $\underline{z}(k)$ depends only on $\underline{\omega}(k-1), \underline{\omega}(k-2), \dots, \underline{\omega}(1)$; hence $Z(k+1)$ depends on $\underline{\omega}(k)$, but $Z(k)$ depends at most on $\underline{\omega}(k-1)$. The single-stage predicted estimate, $\hat{\underline{x}}(k+1|k)$, *in Eq. (9) can be obtained directly* from Eq. (1) by operating on both sides of the state equation with the conditional expectation operator $E\{\cdot|Z(k)\}$. The term $E\{\underline{\omega}(k)|Z(k)\} = E\{\underline{\omega}(k)\} = \underline{0}$ by virtue of our preceding remark and the fact that $\underline{\omega}(k)$ is a zero mean process.

In order to obtain the first meaningful minimum variance estimate of $\underline{\omega}(k)$, operate on both sides of the state equation in Eq. (1) with the operator $E\{\cdot|Z(k+1)\}$;

$$\begin{aligned} E\{\underline{x}(k+1)|Z(k+1)\} &= \Phi E\{\underline{x}(k)|Z(k+1)\} \\ &\quad + E\{\underline{\omega}(k)|Z(k+1)\} \end{aligned} \tag{15}$$

The last term in Eq. (15) will not be zero because $Z(k+1)$ depends on $\underline{\omega}(k)$; hence, we obtain the following estimator for $\underline{\omega}(k)$:

$$\hat{\underline{\omega}}(k|k+1) = \hat{\underline{x}}(k+1|k+1) - \Phi\hat{\underline{x}}(k|k+1) \tag{16}$$

Observe that this is a single-stage smoothed estimate of $\underline{\omega}(k)$ which requires not only the optimal filtered estimate of $\underline{x}(k+1)$, but also the optimal single-stage smoothed estimate of $\underline{x}(k)$, for its implementation.

By way of these calculations we have established the fact that <u>minimum variance estimates of $\underline{\omega}(k)$ will be optimal smoothed estimates.</u> In the remaining paragraphs of this section we shall explore $\hat{\underline{\omega}}(k|k+1)$ and its estimation error in more detail, obtain recursive equations for an optimal ℓ-stage smoothed estimator, $\hat{\underline{\omega}}(k|k+\ell)$, and its error covariance matrix, and, shall obtain comparable results for estimates of $\underline{w}(k)$ in $\mathscr{S}_2$, Eq. (2).

<u>Single-Stage Optimal Smoothed Estimates of $\underline{\omega}(k)$</u>

Here we develop a more useful equation for $\hat{\underline{\omega}}(k|k+1)$ *then Eq. (16)*, and develop an equation for the error covariance matrix $\psi_{\omega}(k|k+1)$, where

$$\psi_{\omega}(k|k+1) \triangleq E\{[\tilde{\underline{\omega}}(k|k+1) - E\{\tilde{\underline{\omega}}(k|k+1)\}][\text{same}]'\} \tag{17}$$

<u>Theorem 1.</u> For system $\mathscr{S}_1$,

$$\hat{\underline{\omega}}(k|k+1) = QP^{-1}(k+1|k)K(k+1)\tilde{\underline{z}}(k+1|k) \tag{18}$$

and

$$\psi_{\omega}(k|k+1) = Q - QH'[HP(k+1|k)H' + R]^{-1}HQ \tag{19}$$

Proof: (a) <u>Derivation of Eq. (18).</u> Meditch (12) derives the following equation for $\hat{\underline{x}}(k|k+1)$:

$$\hat{\underline{x}}(k|k+1) = \hat{\underline{x}}(k|k) + A(k)K(k+1)\tilde{\underline{z}}(k+1|k) \tag{20}$$

where

$$A(k) = P(k|k)\Phi'P^{-1}(k+1|k) \tag{21}$$

Substitute Eqs. (20) and (13) into Eq. (16), making use of Eq. (9), to show that

$$\hat{\underline{\omega}}(k|k+1) = [I - \Phi A(k)]K(k+1)\tilde{\underline{z}}(k+1|k) \tag{22}$$

From Eqs. (21) and (10), it follows that

$$I - \Phi A(k) = QP^{-1}(k+1|k) \tag{23}$$

Substitute Eq. (23) into Eq. (22) in order to obtain the desired expression for $\hat{\underline{\omega}}(k|k+1)$ in Eq. (18).

(b) <u>Derivation of Eq. (19).</u> Signal $\tilde{\underline{z}}(k+1|k)$, the innovations process, is known to be zero mean and white (13), and,

$$E\{\tilde{\underline{z}}(k+1|k)\tilde{\underline{z}}'(k+1|k)\} = HP(k+1|k)H' + R. \tag{24}$$

Subsequently, $E\{\hat{\underline{\omega}}(k|k+1)\} = 0$, which means that $E\{\tilde{\underline{\omega}}(k|k+1)\} = E\{\underline{\omega}(k)\} - E\{\hat{\underline{\omega}}(k|k+1)\} = \underline{0}$, and therefore, that

$$\begin{aligned} \psi_{\omega}(k|k+1) &= E\{\tilde{\underline{\omega}}(k|k+1)\tilde{\underline{\omega}}'(k|k+1)\} \\ &= E\{[\underline{\omega}(k) - \hat{\underline{\omega}}(k|k+1)][\underline{\omega}(k) - \hat{\underline{\omega}}(k|k+1)]'\} \\ &= Q - E\{\underline{\omega}(k)\hat{\underline{\omega}}'(k|k+1)\} - E\{\hat{\underline{\omega}}(k|k+1)\underline{\omega}'(k)\} \\ &\quad + E\{\hat{\underline{\omega}}(k|k+1)\hat{\underline{\omega}}'(k|k+1)\} \end{aligned} \tag{25}$$

From Eqs. (18), (11), and (24), it follows that

$$E\{\hat{\underline{\omega}}(k|k+1)\hat{\underline{\omega}}'(k|k+1)\} = QH'[HP(k+1|k)H' + R]^{-1}HQ \tag{26}$$

Next, we shall show that

$$E\{\underline{\omega}(k)\tilde{\underline{z}}'(k+1|k)\} = QH'. \tag{27}$$

Recall, that

$$\underline{\tilde{z}}(k+1|k) = H\underline{\tilde{x}}(k+1|k) + \underline{v}(k+1) \tag{28}$$

where $\underline{\tilde{x}}(k+1|k) = \underline{x}(k) - \underline{\hat{x}}(k+1|k)$. Meditch (12) derives the following equations for $\underline{\tilde{x}}(k+1|k)$ and $\underline{\tilde{x}}(k|k)$ [Eqs. (5.63) and (5.64), respectively in 12]:

$$\underline{\tilde{x}}(k+1|k) = \Phi\underline{\tilde{x}}(k|k) + \underline{\omega}(k) \tag{29}$$

and

$$\underline{\tilde{x}}(k|k) = [I - K(k)H]\Phi\underline{\tilde{x}}(k-1|k-1) + [I - K(k)H]\underline{\omega}(k-1) - K(k)\underline{v}(k) \tag{30}$$

Substitute Eq. (29) into Eq. (28), to show that

$$\underline{\tilde{z}}(k+1|k) = H\Phi\underline{\tilde{x}}'(k|k) + H\underline{\omega}(k) + \underline{v}(k+1) \tag{31}$$

Observe, from Eq. (30), that $\underline{\tilde{x}}(k|k)$ does not depend on $\underline{\omega}(k)$; it depends at most on $\underline{\omega}(k-1)$; hence,

$$\begin{aligned} E\{\underline{\omega}(k)\underline{\tilde{z}}'(k+1|k)\} &= E\{\underline{\omega}(k)\underline{\tilde{x}}'(k|k)\}(H\Phi)' \\ &\quad + QH' + E\{\underline{\omega}(k)\underline{v}'(k+1)\} \\ &= QH' \end{aligned} \tag{32}$$

which is Eq. (27).

Finally, to obtain $\psi_{\omega}(k|k+1)$ in Eq. (19), substitute Eqs. (26), (18), and (32) into Eq. (25). □

Comment 1. It is interesting to observe, from Eq. (18), that the estimator of white noise process $\underline{\omega}(k)$, $\underline{\hat{\omega}}(k|k+1)$, is itself a white noise process. This is especially useful for the predictive deconvolution problem that was described in Section I, where the reflection coefficient sequence, which is to be estimated, is white. Additionally, we observe from Eq. (19) that $\psi_{\omega}(k|k+1) \leq Q$, which means that we can expect to do better by using $\underline{\hat{\omega}}(k|k+1)$ then by using a zero estimator for $\underline{\omega}(k)$.

ℓ-Stage Optimal Smoothed Estimates of $\underline{\omega}(k)$

Here we generalize the results of the preceding paragraph, obtaining $\underline{\hat{\omega}}(k|k+\ell)$ and $\psi_{\omega}(k|k+\ell)$. These estimator equations are useful in that they permit us to compute fixed-interval, fixed-point, or fixed-lag smoothed estimates of the plant noise. We define $\psi_{\omega}(k|k+\ell)$ as follows:

$$\psi_{\omega}(k|k+\ell) = E\{[\underline{\tilde{\omega}}(k|k+\ell) - E\{\underline{\tilde{\omega}}(k|k+\ell)\}][\text{same}]'\} \tag{33}$$

where

$$\underline{\tilde{\omega}}(k|k+\ell) = \underline{\omega}(k) - \underline{\hat{\omega}}(k|k+\ell) \tag{34}$$

Theorem 2. For system $\mathscr{S}_1$,

$$\underline{\hat{\omega}}(k|k+\ell) = \underline{\hat{\omega}}(k|k+\ell-1) + N(k|k+\ell)\underline{\tilde{z}}(k+\ell|k+\ell-1) \tag{35}$$

where $\ell = 1, 2, \ldots, \underline{\hat{\omega}}(k|k) \triangleq \underline{0}$,

$$N(k|k+1) = QP^{-1}(k+1|k)K(k+1) \tag{36}$$

$$N(k|k+\ell) = QP^{-1}(k+1|k)\prod_{i=k+1}^{k+\ell-1} A(i)K(k+\ell) \tag{37}$$

and[3]

$$A(i) = P(i|i)\Phi'P^{-1}(i+1|i) \tag{38}$$

Additionally,

$$\psi_{\omega}(k|k+\ell) = \psi_{\omega}(k|k+\ell-1) - N(k|k+\ell)[HP(k+\ell|k+\ell-1)H' + R]N'(k|k+\ell) \tag{39}$$

where $\psi_{\omega}(k|k) \triangleq Q$.

Comment 2. Assuming the truth of Eq. (35), then another representation for $\underline{\hat{\omega}}(k|k+\ell)$, which is nonrecursive, is

$$\underline{\hat{\omega}}(k|k+\ell) = \sum_{j=1}^{\ell} N(k|k+j)\underline{\tilde{z}}(k+j|k+j-1) \tag{40}$$

This result is obtained by iterating Eq. (35) on ℓ. □

Comment 3. Suppose we have a complete set of measurements, $\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(L)$, available. Then we can use all of these measurements to obtain fixed-interval estimates of $\underline{\omega}(0), \underline{\omega}(1), \ldots, \underline{\omega}(L-1)$. In Equation (40), set $k+\ell = L$ and let $\ell = 1, 2, \ldots, L$ to obtain the following fixed-interval estimator of plant noise:

$$\underline{\hat{\omega}}(L-\ell|L) = \sum_{j=1}^{\ell} N(L-\ell|L-\ell+j)\underline{\tilde{z}}(L-\ell+j|L-\ell+j-1) \tag{41}$$

A fixed-point estimator of $\underline{\omega}(k)$ is $\underline{\hat{\omega}}(k|k+\ell)$ where k is fixed and ℓ is varied. Our formulas in Theorem 2 are in fixed-point format, and can be used to enhance estimates at specific values of k.

A fixed-lag estimator of $\underline{\omega}(k)$ is $\underline{\hat{\omega}}(k|k+\ell)$ where ℓ is fixed and k is varied. As such, $\underline{\hat{\omega}}(k|k+\ell)$ utilizes the window of measurements, $\underline{z}(k+1), \underline{z}(k+2), \ldots, \underline{z}(k+\ell)$. For example, $\underline{\hat{\omega}}(1|1+\ell)$ utilizes measurements $\underline{z}(2), \underline{z}(3), \ldots, \underline{z}(\ell+1)$, whereas $\underline{\hat{\omega}}(2|2+\ell)$ utilizes the measurements $\underline{z}(3), \underline{z}(4), \ldots, \underline{z}(\ell+2)$. A fixed-lag estimate of $\underline{\omega}(k)$ would be useful in those situations where we decide to use only a fixed length of data to obtain the optimal estimate. □

Proof of Theorem 2: (a) Derivation of Eq. (35). Let

$$Z(k+\ell-1) = \{\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(k+\ell-1)\} \tag{42}$$

and

$$Z(k+\ell) = \{\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(k+\ell)\} \tag{43}$$

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[3] $\prod_{i=1}^{I} A(i) = A(1)A(2)\ldots A(I-1)A(I)$.

Operate on both sides of the state equation in Eq. (1) with operators $E\{\cdot|Z(k+\ell-1)\}$ and $E\{\cdot|Z(k+\ell)\}$ to obtain the following results:

$$\hat{\underline{\omega}}(k|k+\ell-1)=\hat{\underline{x}}(k+1|k+\ell-1)-\Phi\hat{\underline{x}}(k|k+\ell-1) \tag{44}$$

and

$$\hat{\underline{\omega}}(k|k+\ell)=\hat{\underline{x}}(k+1|k+\ell)-\Phi\hat{\underline{x}}(k|k+\ell) \tag{45}$$

Meditch (12) derives the following equation [Eq. (6.60) in 12] for $\hat{\underline{x}}(k|j)$, $j=k+1, k+2, \ldots$:

$$\hat{\underline{x}}(k|j)=\hat{\underline{x}}(k|j-1)+M(k|j)\tilde{\underline{z}}(j|j-1) \tag{46}$$

where

$$M(k|j)=\left[\prod_{i=k}^{j-1}A(i)\right]K(j) \tag{47}$$

We use Eq. (46) to obtain expressions for $\hat{\underline{x}}(k|k+\ell)$ and $\hat{\underline{x}}(k+1|k+\ell)$, as follows:

$$\hat{\underline{x}}(k|k+\ell)=\hat{\underline{x}}(k|j)\Big|_{j=k+\ell} \tag{48}$$

and

$$\hat{\underline{x}}(k+1|k+\ell)=\hat{\underline{x}}(k|j)\Big|_{k=k+1,\ j=k+\ell} \tag{49}$$

Substitute Eqs. (48) and (49), using Eqs. (46), (47), and (44), into Eq. (45) to show that

$$\hat{\underline{\omega}}(k|k+\ell)=\hat{\underline{\omega}}(k|k+\ell-1)+[I-\Phi A(k)]\prod_{i=k+1}^{k+\ell-1}A(i)$$

$$*K(k+\ell)\tilde{\underline{z}}(k+\ell|k+\ell-1) \tag{50}$$

Apply Eq. (23) to Eq. (50), and define matrix $N(k|k+\ell)$ as in Eq. (37), to obtain our Eq. (35). We define $N(k|k)$ as in Eq. (36) so as to be able to obtain $\hat{\underline{\omega}}(k|k+1)$ from Eq. (35) that is consistent with our earlier results in Theorem 1.

(b) Derivation of Eq. (39). From Eq. (40), we see that $E\{\hat{\underline{\omega}}(k|k+\ell)\}=\underline{0}$; subsequently, $E\{\tilde{\underline{\omega}}(k|k+\ell)\}=\underline{0}$, and

$$\psi_{\omega}(k|k+\ell)=E\{\tilde{\underline{\omega}}(k|k+\ell)\tilde{\underline{\omega}}'(k|k+\ell)\} \tag{51}$$

where, from Eqs. (34) and (35),

$$\tilde{\underline{\omega}}(k|k+\ell)=\tilde{\underline{\omega}}(k|k+\ell-1)-N(k|k+\ell)\tilde{\underline{z}}(k+\ell|k+\ell-1) \tag{52}$$

Substitute Eq. (52) into Eq. (51), using Eq. (24) for $k=k+\ell$, to obtain the following expression for $\psi_{\omega}(k|k+\ell)$:

$$\psi_{\omega}(k|k+\ell)=\psi_{\omega}(k|k+\ell-1)+N(k|k+\ell)[HP(k+\ell|k+\ell-1)H'$$

$$+R]N(k|k+\ell) \tag{53}$$

$$-N(k|k+\ell)E\{\tilde{\underline{z}}(k+\ell|k+\ell-1)\tilde{\underline{\omega}}'(k|k+\ell-1)\}$$

$$-E\{\tilde{\underline{\omega}}(k|k+\ell-1)\tilde{\underline{z}}'(k+\ell|k+\ell-1)\}N'(k|k+\ell)$$

Because $\tilde{\underline{z}}$ is a white sequence,

$$E\{\tilde{\underline{z}}(k+j|k+j-1)\tilde{\underline{z}}'(k+\ell|k+\ell-1)\}=0$$

for all $j\neq\ell$; hence, using Eqs. (40) and (34), it is straightforward to show that

$$E\{\tilde{\underline{\omega}}(k|k+\ell-1)\tilde{\underline{z}}'(k+\ell|k+\ell-1)\}$$

$$=E\{\underline{\omega}(k)\tilde{\underline{z}}'(k+\ell|k+\ell-1)\} \tag{54}$$

The quantity on the right-hand side of Eq. (54) is evaluated in Appendix A, as $\psi'_{\tilde{z}\omega}(\ell|\ell-1)$:

$$\psi'_{\tilde{z}\omega}(\ell|\ell-1)=N(k|k+\ell)[HP(k+\ell|k+\ell-1)H'+R] \tag{55}$$

Apply Eqs. (54) and (55) to Eq. (53) to obtain our results in Eq. (39).

Optimal Smoothed Estimates of $\underline{w}(k)$

We now direct our attention at estimation of $\underline{w}(k)$ in $\mathscr{S}_2$, Eq. (2), when $\underline{w}\in R^q$, $q\leq n$. Signal $\underline{w}(k)$ is white with covariance Q_1.

Let $\hat{\underline{w}}(k|k+\ell)$ denote the minimum-variance ℓ-stage smoothed estimator of $\underline{w}(k)$, and $\psi_w(k|k+\ell)$ its error covariance matrix,

$$\psi_w(k|k+\ell)=E\{[\hat{\underline{w}}(k|k+\ell)-E\{\hat{\underline{w}}(k|k+\ell)\}][\text{same}]'\} \tag{56}$$

Theorem 3. For System $\mathscr{S}_2$,

$$\hat{\underline{w}}(k|k+\ell)=\hat{\underline{w}}(k|k+\ell-1)+N_w(k|k+\ell)\tilde{\underline{z}}(k+\ell|k+\ell-1) \tag{57}$$

where $\ell=1,2,\ldots$, $\hat{\underline{w}}(k|k)\triangleq\underline{0}$,

$$N_w(k|k+1)=Q_1\Gamma'P^{-1}(k+1|k)K(k+1) \tag{58}$$

$$N_w(k|k+\ell)=Q_1\Gamma'P^{-1}(k+1|k)\prod_{i=k+1}^{k+\ell-1}A(i)K(k+\ell) \tag{59}$$

and $A(i)$ is defined in Eq. (38). Additionally,

$$\psi_w(k|k+\ell)=\psi_w(k|k+\ell-1)-N_w(k|k+\ell)[HP(k+\ell|k+\ell-1)H'$$

$$+R]N'_w(k|k+\ell) \tag{60}$$

where $\psi_w(k|k)\triangleq Q_1$.

Proof: System $\mathscr{S}_1$, in Eq. (1), can be made equivalent to system $\mathscr{S}_2$, in Eq. (2), by setting $\underline{\omega}(k)=\Gamma\underline{w}(k)$, in which case

$$\hat{\underline{\omega}}(k|k+\ell)=\Gamma\hat{\underline{w}}(k|k+\ell) \tag{61}$$

and

$$Q=\Gamma Q_1\Gamma' \tag{62}$$

All of the results in Theorem 3 follow upon substitution of Eqs. (61) and (62) into the respective Theorem 2 equations. □

Comment 4. In the reflection seismology problem described in Section I [see Eqs. (6) and (7)], $\underline{w}(k)$ is a scalar --- the reflection coefficient sequence, $\mu(k)$; hence, the results in Theorem 3 are the ones which are applicable to that problem.

ZERO MARKOV PARAMETERS CASE

Ott and Meder (9) consider an example which they state "... is of special interest for seismic exploration ...". *It is a one dimensional damped harmonic oscillator which is excited by impulses of random intensity in random time instances.* For their example[4],

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1 & T \\ -\beta T & 1-\alpha T \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathfrak{w}_E T \end{pmatrix} \mu(k) \tag{63}$$

$$z(k) = (1 \quad 0) \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + v(k) \tag{64}$$

If one substitutes the Φ, Γ, and H matrices for this example into Eqs. (57) and (58), for $\ell = 1$, he finds that $\hat{\mu}(k|k+1) = 0$. Additionally, $\hat{\mu}(k|k+\ell) \neq 0$ for $\ell \geq 2$. This result suggests that perhaps the *structures of Φ, Γ, and H establish a first value of ℓ for which $\underline{\hat{w}}(k|k+\ell) \neq \underline{0}$.*

Let us take a closer look at $N_w(k|k+1)$ in Eq. (58). Substitute Eq. (11) into Eq. (58), to show that

$$N_w(k|k+1) = Q_1(H\Gamma)'[HP(k+1|k)H' + R]^{-1} \tag{65}$$

Observe that $N_w(k|k+1)$ depends on the Markov parameter $H\Gamma$ (15). In the Ott/Meder example $H\Gamma = 0$.

We generalize the preceding observation in the following theorem, the proof of which is given in Appendix B.

Theorem 4. *For system $\mathcal{S}_2$, if $H\Gamma = H\Phi\Gamma = \dots = H\Phi^{j-1}\Gamma = 0$, and $H\Phi^j\Gamma \neq 0$, then*

$$\underline{\hat{w}}(k|k+\ell) = \underline{0} \quad \text{for} \quad \ell = 1, 2, \dots, j \tag{66}$$

and

$$\underline{\hat{w}}(k|k+j+1) \neq 0 \ . \tag{67}$$

This theorem states that if the first j Markov parameters are zero, then the first non-zero estimator of $\underline{\hat{w}}(k)$ is the one which looks $j+1$ points into the future.

Some consequences of Theorem 4 are given in the following corollaries to that theorem.

---

[4] Equation (63) is obtained by discretizing the equation $\ddot{x}(t)+\alpha\dot{x}(t)+\beta x(t)=\mathfrak{w}_E\mu(t)$, assuming that $t_{k+1}-t_k = T$ is small.

Corollary 1. Under the conditions of Theorem 4,

$$\Psi_w(k|k+\ell) = Q_1 \quad \text{for} \quad \ell = 1, 2, \dots, j \tag{68}$$

The proof of this simple result follows directly from the fact that, because of Eq. (66), $\underline{\tilde{w}}(k|k+\ell) = \underline{w}(k)$ for $\ell = 1, 2, \dots, j$.

Corollary 2. Under the conditions of Theorem 4,

$$\Psi_w(k|k+j+1) = Q_1 - Q_1(H\Phi^j\Gamma)'[HP(k+j+1|k+j)H' + R]^{-1}(H\Phi^j\Gamma)Q_1 \tag{69}$$

Proof: From Eqs. (60) and (68), *it follows that*

$$\Psi_w(k|k+j+1) = Q_1 - N_w(k|k+j+1)[HP(k+j+1|k+j)H' + R]N_w'(k|k+j+1) \tag{70}$$

An equation for $N_w(k|k+j+1)$ is developed in Appendix B, Eq. (B-10). Substitute that equation into Eq. (70) to obtain the desired result in Eq. (69).

AN AD HOC ESTIMATOR

Ott and Meder (9) define a "prediction error filter" for $\underline{\mathfrak{w}}(k)$, in system $\mathcal{S}_1$, as the difference between $\underline{\hat{x}}(k+1|k+1)$ and $\underline{\hat{x}}(k+1|k)$. In this section, we shall examine their estimator and compare its performance with our estimator $\underline{\hat{\mathfrak{w}}}(k|k+1)$(Section II). We shall also study the consequences of a prediction error filter for estimating $\underline{w}(k)$ in system $\mathcal{S}_2$. As such, this section serves to illustrate some of our theoretical results.

Ad Hoc Estimator of $\underline{\mathfrak{w}}(k)$

Combine Eqs. (13) and (9), to show that

$$\underline{\hat{x}}(k+1|k+1) = \Phi\underline{\hat{x}}(k|k) + K(k+1)\underline{\tilde{z}}(k+1|k) \tag{71}$$

in which we recall that $\underline{\tilde{z}}(k+1|k)$ is the white innovations process. Comparing the structures of Eq. (71) and the state equation in Eq. (1), we define a "prediction error filter" for $\underline{\mathfrak{w}}(k)$, denoted here as $\underline{\hat{\mathfrak{w}}}_1(k)$, as

$$\underline{\hat{\mathfrak{w}}}_1(k) \triangleq K(k+1)\underline{\tilde{z}}(k+1|k) \tag{72}$$

Observe that $\underline{\hat{\mathfrak{w}}}_1(k)$ is a white estimator of $\underline{\mathfrak{w}}(k)$, and that $\underline{\hat{\mathfrak{w}}}_1(k) = \underline{\hat{x}}(k+1|k+1) - \underline{\hat{x}}(k+1|k)$. Observe, also, that $\underline{\hat{\mathfrak{w}}}(k)$ is not an optimal estimator of $\underline{\mathfrak{w}}(k)$; it has merely been defined as in Eq. (72). As such, we refer to the prediction error filter as an ad hoc estimator.

We shall compare $\underline{\hat{\mathfrak{w}}}_1(k)$ with $\underline{\hat{\mathfrak{w}}}(k|k+1)$. Let $\Psi_{\mathfrak{w}1}(k)$ denote the error covariance matrix for the ad hoc estimator $\underline{\hat{\mathfrak{w}}}_1(k)$; i.e.,

$$\Psi_{\mathfrak{w}1}(k) = E\{[\underline{\tilde{\mathfrak{w}}}_1(k) - E\{\underline{\tilde{\mathfrak{w}}}_1(k)\}][\text{same}]'\} \tag{73}$$

where

$$\tilde{\underline{w}}_1(k) = \underline{w}(k) - \hat{\underline{w}}_1(k) \tag{74}$$

From Eq. (72), it follows that $E\{\hat{\underline{w}}_1(k)\} = \underline{0}$; hence, $E\{\tilde{\underline{w}}_1(k)\} = \underline{0}$, so that

$$\psi_{w1}(k) = E\{\tilde{\underline{w}}_1(k)\tilde{\underline{w}}_1'(k)\} \tag{75}$$

Substitute Eqs. (74) and (72) into Eq. (75), using Eqs. (24), (27), and (11) to show that

$$\psi_{w1}(k) = Q + PH'(HPH'+R)^{-1}HP - QH'(HPH'+R)^{-1}HP - PH'(HPH'+R)^{-1}HQ \tag{76}$$

in which P is short for $P(k+1|k)$.

<u>Theorem 5.</u> For system $\mathscr{S}_1$,

$$\psi_w(k|k+1) \le \psi_{w1}(k) \tag{77}$$

Proof: From Eqs. (76) and (19), it follows that

$$\psi_{w1}(k) - \psi_w(k|k+1) = (P-Q)[H'(HPH'+R)^{-1}H](P-Q) \tag{78}$$

Matrix $R > 0$, and $(HPH'+R) > 0$ for the inverse of $(HPH'+R)$, which is used to calculate $K(k+1)$, to exist. It is straightforward to show, therefore, that the right-hand side of Eq. (78) is positive semi-difinite; hence, the truth of Eq. (77). □

The significance of Eq. (77) is that it implies that one will usually obtain better performance with our single-stage estimator, $\hat{\underline{w}}(k|k+1)$, then with Ott and Meder's ad hoc prediction error filter; and, that one cannot do worse using our estimator rather than the prediction error filter.

<u>Ad Hoc Estimator of $\underline{w}(k)$</u>

Now compare the structures of Eq. (71) and the state equation in Eq. (2), to obtain the following "correspondence":

$$\Gamma\hat{\underline{w}}_1(k) \triangleq K(k+1)\tilde{\underline{z}}(k+1|k) \tag{79}$$

In this equation, $\hat{\underline{w}}_1(k)$ denotes the prediction error filter for $\underline{w}(k)$. Unfortunately, Eq. (79) is an overdetermined system of equations; i.e., dim $\Gamma\hat{\underline{w}}_1(k) = n$ and $n > \dim \hat{\underline{w}}_1(k) = q$. We use the pseudo-inverse of Γ, $(\Gamma'\Gamma)^{-1}\Gamma'$, to obtain the following ad hoc prediction error filter for $\underline{w}(k)$:

$$\hat{\underline{w}}_1(k) \triangleq (\Gamma'\Gamma)^{-1}\Gamma' K(k+1)\tilde{\underline{z}}(k+1|k) \tag{80}$$

Let $\psi_{w1}(k)$ denote the error covariance matrix for $\hat{\underline{w}}_1(k)$, defined analogously to $\psi_{w1}(k)$ in Eq. (73). Proceeding as we did for the calculation of $\psi_{w1}(k)$ in Eq. (76), it is straightforward to show that

$$\psi_{w1}(k) = Q_1 + (\Gamma'\Gamma)^{-1}\Gamma'PH'(HPH'+R)^{-1}HP\Gamma(\Gamma'\Gamma)^{-1} - Q_1\Gamma'H'(HPH'+R)^{-1}HP\Gamma(\Gamma'\Gamma)^{-1} - (\Gamma'\Gamma)^{-1}\Gamma'PH'(HPH'+R)^{-1}H\Gamma Q_1 \tag{81}$$

where, $P = P(k+1|k)$.

<u>*Theorem 6.*</u> For system $\mathscr{S}_2$,

$$\psi_w(k|k+1) \le \psi_{w1}(k) \tag{82}$$

The proof of this result parallels the proof of Theorem 5, and is *left to the reader.* Once again, we see that we can expect to do at least as good using our single-stage smoother, $\hat{\underline{w}}(k|k+1)$, instead of the prediction error filter, $\hat{\underline{w}}_1(k)$.

Finally, let us reconsider the harmonic oscillator example considered by Ott and Meder, whose equations are given in Eqs. (63) and (64). Recall, from our discussions in Section III, that $H\Gamma = 0$ for their system. Under this condition, $\psi_{w1}(k)$ in Eq. (81) reduces to

$$\psi_{w1}(k) = Q_1 + (\Gamma'\Gamma)^{-1}\Gamma'PH'(HPH'+R)^{-1}HP\Gamma(\Gamma'\Gamma)^{-1} \tag{83}$$

which means that, for their example,

$$\psi_{w1}(k) \ge Q_1 \tag{84}$$

From an *estimation error-covariance point of* view, Eq. (85) implies that the prediction error filter performs no better, and usually worse, than the zero estimator of $\underline{w}(k)$! Why bother with $\hat{\underline{w}}_1(k)$, when $\underline{0}$ appears to be a better estimator of $\underline{w}(k)$? The answer to this question is related to the seismic data processing problem described in Section I. We are interested in a time-series for the reflection coefficient sequence. By examining peak values and their times of occurrence, we can infer some things about subsurface geometry. Even a time-series estimate such as $\hat{w}_1(k)$ will have peak values (*for low signal to noise ratios*) that make $\hat{\underline{w}}_1(k)$ useful [*see* <u>9</u> *for some* numerical examples that support this statement]; whereas, a zero estimate of $\hat{\underline{w}}(k)$ does not contain any of this useful information.

CONCLUSIONS

We have demonstrated how an important application can be interpreted, from a state equation point of view, as one of estimating plant noise. Our emphasis in this paper has been on the theoretical development of white, plant noise estimators. We have shown that such estimators provide us with smoothed estimates of the plant noise. By means of the equations which are derived in the main body of this paper, it is possible to obtain fixed-interval, fixed-point, or fixed-lag optimal smoothed estimates of white plant noise, as well as respective error covariance-matrix information.

We have also compared our optimal estimators with an ad hoc "prediction error filter," and have shown that better performance can be attained with our estimators.

Much work remains to be done, including application of our smoothers to real seismic data; study of computational requirements of our smoothers; development of fast realizations of our smoothing equations; and the extension to multi-input multi-output seismic data processing problems. Recall, also, that a major feature that the Kalman filter has to offer over the Wiener filter is it can be used to estimate states in time-varying systems. Additionally, it can be "extended" to estimate states in nonlinear systems. The extensions of the results in this paper to estimation of the reflection coefficient sequence for time-varying and nonlinear source-earth models are other areas for additional work.

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REFERENCES

1 Wood, L.C. and Treitel, S., "Seismic Signal Processing," Proc. of the IEEE, Vol. 63, No. 4, April 1975, pp. 649-661.

2 Anstey, N.A., "Seismic Prospecting Instruments, Vol. 1: Signal Characteristics and Instrument Specifications," 1970, Gebruder Borntraeger, Berlin.

3 Dix, C.H., "Seismic Prospecting for Oil," 1952, Harper Brothers, New York.

4 Robinson, E.A., "Predictive Decomposition of Seismic Traces," Geophysics, Vol. 22, No. 4, October 1957, pp. 767-778.

5 Robinson, E.A., "Predictive Decomposition of Time Series with Application to Seismic Exploration," Geophysics, Vol. 32, No. 3, June 1967, pp. 418-484 (This is a reprint of Robinson's 1954 Ph.D. thesis).

6 Ricker, N., "The Form and Nature of Seismic Waves and the Structure of Seismograms," Geophysics, Vol. 5, 1940, pp. 348-366.

7 Bayliss, J.W. and Brigham, E.O., "Application of the Kalman Filter to Continuous Signal Restoration," Geophysics, Vol. 35, No. 1, February 1970, pp. 2-23.

8 Crump, N., "A Kalman Filter Approach to the Deconvolution of Seismic Signals," Geophysics, Vol. 39, No. 1, February 1974, pp. 1-13.

9 Ott, N. and Meder, H.G., "The Kalman Filter as a Prediction Error Filter," Geophysical Prospecting, Vol. 20, 1972, pp. 549-560.

10 Schwarz, R.J. and Friedland, B., "Linear Systems," 1965, McGraw-Hill, New York.

11 Chen, C.T., "Introduction to Linear System Theory," 1970, Holt-Rinehart and Winston, New York.

12 Meditch, J.S., "Stochastic Optimal Linear Estimation and Control," 1969, McGraw-Hill, New York.

13 Kailath, T., "An Innovations Approach to Least-Squares Estimation - Part I: Linear Filtering in Additive White Noise," IEEE Transactions on Automatic Control, Vol. AC-13, December 1968, pp. 646-655.

14 Mendel, J.M., "Discrete Techniques of Parameter Estimation: The Equation Error Formulation," 1973, Marcel Dekker, Inc., New York.

15 Ho, B.L. and Kalman, R.E., "Effective Construction of Linear State-Variable Models from Input/Output Functions," Regelungstechnik, Vol. 12, 1966, pp. 545-548.

APPENDIX A

Calculation of $E\{\underline{\tilde{z}}(k+\ell|k+\ell-1)\underline{w}'(k)\}$

For notational convenience, let

$$E\{\underline{\tilde{z}}(k+\ell|k+\ell-1)\underline{w}'(k)\} \overset{\Delta}{=} \psi_{\tilde{z}w}(\ell|\ell-1) \tag{A-1}$$

Theorem. For system $\mathscr{S}_1$,

$$\psi_{\tilde{z}w}(1|0) = HQ \tag{A-2}$$

$$\psi_{\tilde{z}w}(\ell|\ell-1) = H\Phi\Phi^*(k+\ell-1,k)\Phi^{-1}Q \tag{A-3}$$

where

$$\Phi^*(k+\ell-1,k) = \prod_{j=\ell-1}^{1} \{[I-K(k+j)H]\Phi\} \tag{A-4}$$

and $\ell = 2, 3, \ldots$. Additionally,

$$\psi'_{\tilde{z}w}(\ell|\ell-1) = N(k|k+\ell)[HP(k+\ell|k+\ell-1)H'+R] \tag{A-5}$$

where $N(k|k+\ell)$ is defined in Eqs. (36) and (37).

Proof: (a) Derivation of Eq. (A-2)(ℓ=1 case).

$$\psi_{\tilde{z}w}(1|0) = E\{\underline{\tilde{z}}(k+1|k)\underline{w}'(k)\} \tag{A-6}$$

The truth of Eq. (A-2) now follows from our developments in Eqs. (27) through (32), since $\psi_{\tilde{z}w}(1|0)$ is simply the transpose of the results in Eq. (27).

(b) Derivation of Eqs. (A-3) and (A-4). Set $k = k+\ell-1$ in Eq. (31) to show that

$$\underline{\tilde{z}}(k+\ell|k+\ell-1) = H\Phi\underline{\tilde{x}}(k+\ell-1|k+\ell-1) + H\underline{w}(k+\ell-1) + \underline{v}(k+\ell); \tag{A-7}$$

hence,

$$\psi_{\tilde{z}w}(\ell|\ell-1) = H\Phi E\{\underline{\tilde{x}}(k+\ell-1|k+\ell-1)\underline{w}'(k)\} \tag{A-8}$$

since $E\{\underline{w}(k+\ell-1)\underline{w}'(k)\} = 0$ and $E\{\underline{v}(k+\ell)\underline{w}'(k)\} = 0$. Next, write Eq. (30) as

$$\underline{\tilde{x}}(k|k) = \Phi^*(k,k-1)\underline{\tilde{x}}(k-1|k-1) + \Phi^*(k,k-1)\Phi^{-1}\underline{w}(k-1) - K(k)\underline{v}(k) \tag{A-9}$$

where

$$\Phi^*(k, k-1) \stackrel{\Delta}{=} [I - K(k)H]\Phi \quad (A\text{-}10)$$

The solution of Eq. (A-9) for $\tilde{\underline{x}}(k|k)$ is (Chapter 1, 14)

$$\tilde{\underline{x}}(k|k) = \Phi^*(k,0)\tilde{\underline{x}}(0|0) + \sum_{i=1}^{k} \Phi^*(k,i)[\Phi^*(i,i-1)\Phi^{-1}\underline{w}(i-1) - K(i)\underline{v}(i)] \quad (A\text{-}11)$$

where

$$\Phi^*(k,i) = \Phi^*(k,k-1)\Phi^*(k-1,k-2)\ldots\Phi^*(i+1,i) \quad (A\text{-}12)$$

Set $k = k + \ell - 1$ in Eq. (A-11) and observe that the only term which depends on $\underline{w}(k)$ is the one for which $i = k+1$; that term is

$$\Phi^*(k+\ell-1, k+1)\Phi^*(k+1,k)\Phi^{-1}\underline{w}(k) \ ,$$

which can also be written as

$\Phi^*(k+\ell-1,k)\Phi^{-1}\underline{w}(k)$. Based on these results, evaluate the right-hand side of Eq. (A-8) to show that

$$\psi_{\tilde{z}w}(\ell|\ell-1) = H\Phi\Phi^*(k+\ell-1,k)\Phi^{-1}Q \ , \quad (A\text{-}13)$$

which is Eq. (A-3). Substitute Eq. (A-10) into Eq. (A-12) to obtain Eq. (A-4).

(c) Derivation of Eq. (A-5).

Set $k = k + j$ in Eq. (14), to show that

$$I - K(k+j)H = P(k+j|k+j)P^{-1}(k+j|k+j-1). \quad (A\text{-}14)$$

Substitute Eq. (A-14) into Eq. (A-4), take the transpose of the resulting expression, and use Eq. (38) to show that

$$\psi'_{\tilde{z}w}(\ell|\ell-1) = QP^{-1}(k+1|k)A(k+1)A(k+2)\ldots A(k+\ell-2) * P(k+\ell-1|k+\ell-1)\Phi'H' \quad (A\text{-}15)$$

Next, using Eq. (11) in which $k = k+\ell$, write $N(k|k+\ell)$ as

$$N(k|k+\ell) = QP^{-1}(k+1|k)A(k+1)A(k+2)\ldots A(k+\ell-2) * P(k+\ell-1|k+\ell-1)\Phi'H'[HP(k+\ell|k+\ell-1)H' + R]^{-1} \quad (A\text{-}16)$$

Comparing Eqs. (A-15) and (A-16), we see that

$$N(k|k+\ell) = \psi'_{\tilde{z}w}(\ell|\ell-1)[HP(k+\ell|k+\ell-1)H' + R]^{-1}, \quad (A\text{-}17)$$

from which Eq. (A-5) follows.

APPENDIX B

Proof of Theorem 4

The following nonrecursive equation for $\hat{\underline{w}}(k|k+\ell)$ is easily obtained from Eqs. (40), (61), and (59):

$$\hat{\underline{w}}(k|k+\ell) = \sum_{i=1}^{\ell} N_w(k|k+i)\tilde{\underline{z}}(k+i|k+i-1) \quad (B\text{-}1)$$

Lemma. For system $\mathscr{S}_2$, if $H\Gamma = H\Phi\Gamma = \ldots = H\Phi^{q-1}\Gamma = 0$, then $N_w(k|k+q) = 0$.

Proof: From Eq. (50), with $\ell = q$,

$$N_w(k|k+q) = Q_1\Gamma'P^{-1}(k+1|k)\prod_{m=k+1}^{k+q-1} A(m)K(k+q) \quad (B\text{-}2)$$

The following equation (12), which is a well-known alternative to Eq. (14), is used in our proof:

$$P^{-1}(k+1|k) = P^{-1}(k+1|k+1) - H'R^{-1}H \quad (B\text{-}3)$$

Substitute Eq. (B-3) into Eq. (B-2), and assume $H\Gamma = 0$, to show that

$$N_w(k|k+q) = Q_1\Gamma'P^{-1}(k+1|k+1)\prod_{m=k+1}^{k+q-1} A(m)K(k+q) \quad (B\text{-}4)$$

or,

$$N_w(k|k+q) = Q_1\Gamma'P^{-1}(k+1|k+1)A(k+1)\prod_{m=k+2}^{k+q-1} A(m)K(k+q) \quad (B\text{-}5)$$

Substitute Eq. (38), for $i = k+1$, into Eq. (B-5) to show that

$$N_w(k|k+q) = Q_1\Gamma'\Phi'P^{-1}(k+2|k+1)\prod_{m=k+2}^{k+q-1} A(m)K(k+q) \quad (B\text{-}6)$$

Substitute Eq. (B-3), for $k = k+1$, into Eq. (B-6), assume $H\Phi\Gamma = 0$, and, substitute Eq. (38), for $i = k+2$, into the resulting expression, to show that

$$N_w(k|k+q) = Q_1\Gamma'(\Phi^2)'P^{-1}(k+3|k+2)\prod_{m=k+3}^{k+q-1} A(m)K(k+q) \quad (B\text{-}7)$$

Continue this development assuming $H\Phi^2\Gamma = 0$, then $H\Phi^3\Gamma = 0, \ldots$, and then $H\Phi^{q-2}\Gamma = 0$, to show that

$$N_w(k|k+q) = Q_1\Gamma'(\Phi^{q-1})'P^{-1}(k+q|k+q-1)K(k+q) \quad (B\text{-}8)$$

Finally, substitute Eq. (11), for $k = k+q$, into Eq. (B-8) to show that

$$N_w(k|k+q) = Q_1\Gamma'(\Phi^{q-1})'H'[HP(k+q|k+q-1)H' + R]^{-1} \quad (B\text{-}9)$$

If $H\Phi^{q-1}\Gamma = 0$, as assumed, then $N_w(k|k+q) = 0$, which completes the proof of the lemma.

Proof of Theorem 4. Theorem 4 is proved by repeated application of our lemma, as follows. If $H\Gamma = 0$, then $N_w(k|k+1) = 0$, which means that

$\hat{\underline{w}}(k|k+1) = \underline{0}$ [see Eq. (B-1)]. If $H\Gamma = 0$ and $H\Phi\Gamma = 0$, $N_w(k|k+2) = 0$; but, since $H\Gamma = 0$, $N_w(k|k+1) = 0$ as well; hence $\hat{\underline{w}}(k|k+2) = 0$. Proceeding in this manner, we conclude that if $H\Gamma = H\Phi\Gamma = \ldots = H\Phi^{j-1}\Gamma = 0$, then $N_w(k|k+j) = N_w(k|k+j-1) = \ldots = N_w(k|k+2) = N_w(k|k+1) = 0$; hence, $\hat{\underline{w}}(k|k+j) = \underline{0}$. In short, we have shown that, under the conditions of Theorem 4, Eq. (66) is true.

Next, set $q = j+1$ in Eq. (B-9) to show that

$$N_w(k|k+j+1) = Q_1(H\Phi^j\Gamma)'[HP(k+j+1|k+j)H' + R]^{-1} \tag{B-10}$$

Since $H\Phi^j\Gamma \neq 0$, $N_w(k|k+j+1) \neq 0$; hence, under the conditions of the theorem,

$$\hat{\underline{w}}(k|k+j+1) = N_w(k|k+j+1)\tilde{\underline{z}}(k+j+1|k+j) \neq \underline{0}, \tag{B-11}$$

which proves Eq. (67), and, completes the proof.

FIGURES

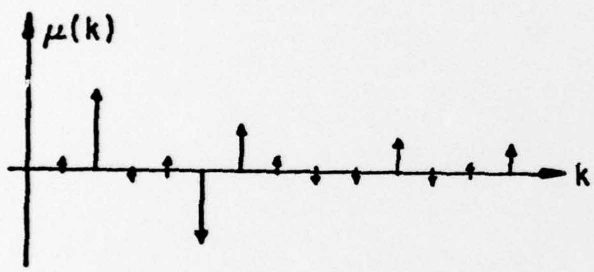

Figure 1. Example of reflection coefficient sequence, $\mu(k)$.

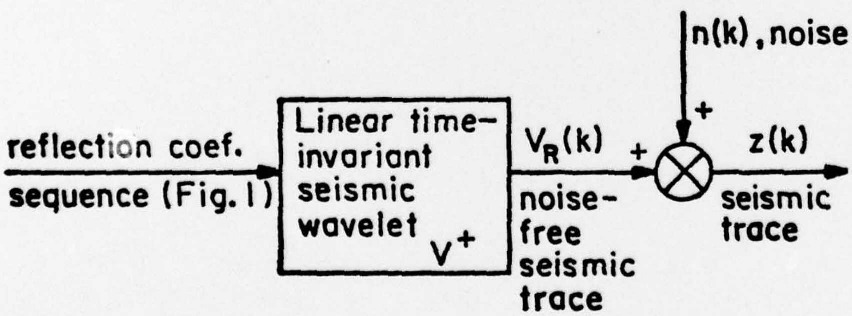

Figure 2. Linear system interpretation for convolution summation model of a seismic trace. Signal $V_R(k)$ is given by Eq. (4).

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This is basically a theoretical paper motivated by a problem from seismic data processing in oil exploration. We develops a Kalman filtering approach to obtaining optimal smoothed estimates of the so-called reflection coefficient sequence. This sequence contains important information about subsurface geometry. Our theoretical problem is one of estimating white plant noise for the systems described in Eqs. (1) and (2). By means of the equations which are derived hereing, it is possible to compute fixed-interval, fixed-point, or fixed-lag optimal smoothed estimates of the reflection coefficient sequence, as well

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20 Abstract

as respective error covariance-matrix information. Our optimal estimators are compared with an ad hoc "prediction error filter," which has recently been reported on in the geophysics literature. Wh show that one of our estimators performs at least as well as, and, in most cases better than, the prediction error filter.

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